Quick Intro to Transpose, Rank, and Linear Dependence, Consistency

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^T = [a \quad b \quad c], \qquad [a \quad b \quad c]^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

Also, familiarize yourself with unique, infinite, and no solution, and consistent/inconsistent:

Unique (Consistent)	**Infinite Solutions (Consistent)**	**No Solution (Inconsistent)**
$\left[\begin{array}{cc\|c} 1 & * & * \\ 0 & 1 & * \end{array}\right]$ Intersection of two lines	$\left[\begin{array}{cc\|c} 1 & * & * \\ 0 & 0 & 0 \end{array}\right]$ Line Multiples	$\left[\begin{array}{cc\|c} 1 & * & * \\ 0 & 0 & 1 \end{array}\right]$ Parallel Lines
$\left[\begin{array}{ccc\|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$ Intersection of 3 planes	$\left[\begin{array}{ccc\|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$ 1 or More Plane Multiples	$\left[\begin{array}{ccc\|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{array}\right]$ Parallel Plane(s)

The matrices above are in REF (Row Echelon Form) and show the solution based on the pivots. Next, we will understand pivots, matrix size and dimensions, what the vectors span, and so on.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ Full,} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ Full Row Rank,} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ Full Column Rank}$$

The rank is equal to the number of pivots or linearly independent columns or the dimension, ...

The size of a matrix is the number of rows and columns—that is, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is 2×3 matrix.

$$A_{2\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

This matrix has rank at most equal to the number of rows.

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(B) = 2 = \dim B \text{. This also means the column vectors span } \mathbb{R}^2.$$

There are two rows and three columns. A set of column vectors is said to be linearly independent or linearly dependent. There are many definitions for this.

Linearly Dependent Set	**Linearly Independent Set**
Possible Column Vectors of B $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \{\mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_3\}$ There are many ways to see this relation...	Row Vectors of B $\{\mathbf{w_1}, \mathbf{w}_2\} = \left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T\right\}$

At this point in Linear Algebra, make sure you drill in your head that every time you are asked a question, it will (in general) lead to the same solution—that is, the answer will just be stated differently, but the method arriving at the answer is the usually same matrix reduction RREF method. I.e., if you are in doubt on an exam, simply put everything in a matrix and solve.

[PL] **Students who do not thoroughly read the Linear Algebra textbook do terribly on** exams because they don't know what the words mean when a true/false section appears. Linear Algebra is more of an English course than a math course.

It should also be noted that whether you are working with vectors, matrices, polynomials, a set of differential equation solutions, or any other set that spans a linear combination is the same concept of arithmetic. E.g., is $\{e^x, e^{-x}, -e^x\}$ a linearly independent set? There are different ways to look at this and define it, but the most common way would be to say that the trivial solution is the only solution to $A\mathbf{x} = \mathbf{0}$—trivial meaning $\mathbf{x} = \mathbf{0}$. **Linearly Dependent**: one or more objects are a multiple or sum of the other objects. **Linearly Independent**: no object is a multiple or sum of the other objects in the set.

NOTE The definitions in the section (outside of the referenced textbooks) are generalized notations and descriptions that are usually universal but may vary course to course.

Show that $\{e^x, e^{-x}, -e^x\}$ is a linearly dependent set.

$$\text{span}\{e^x, e^{-x}, -e^x\} = c_1 e^x + c_2 e^{-x} + c_3(-e^x) = c_1 e^x + c_2 e^{-x} + c_4 e^x$$

$$= (c_1 + c_4)e^x + c_2 e^{-x} = c_5 e^x + c_2 e^{-x} \;\Rightarrow\; \{e^x, e^{-x}, -e^x\} \sim \{e^x, e^{-x}\}.$$

Thus, the set $\{e^x, e^{-x}, -e^x\}$ is a linearly dependent set.

