ODE – Dirac Delta Function and Unit Impulse Function

QUESTION [11] 7.6.7

Solve the second order ODE using method of Laplace Transform,

$$y'' + 2y' = \delta(t-1), \qquad y(0) = 0, \qquad y'(0) = 1.$$

ANSWER

$$y = \frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1)$$

Before we begin investigating this unique function, we must acknowledge that it is an operator and all operators have instructions; without the instructions we cannot perform said operations.

Unit Impulse Function

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \dfrac{1}{2a}, & t_0 - a \le t < t_0 + a, \\ 0, & t \ge t_0 + a \end{cases} \qquad a > 0, t_0 > 0.$$

$$\int_0^\infty \delta_a(t-t_0) = 1.$$

The **Dirac Delta Function**: Defined by taking the limit of the **Unit Impulse** function. That is,

$$\lim_{a\to 0} \delta_a(t-t_0) = \delta_0(t-t_0) \equiv \delta(t-t_0).$$

$$\text{[i]}\ \ \delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \ne t_0 \end{cases}, \qquad \text{[ii]}\ \int_0^\infty \delta(t-t_0)\,dt = 1.$$

FOCUS DEFINITION Dirac Delta Integration

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$

Sifting Property: When an object in the integrand is simply evaluated at the constant within the integrand leading to no formal arithmetic required.

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$

[FE] This formula will be introduced in advance physics related (ENGR) courses junior year.

NOTE All of this is just rules and instructions based on definitions. Don't try to understand anything as it is about as productive as understanding why the letter A has the shape and sound it does—that is, someone defined it that way.

[TM] This is a function that applies in special cases and there is no need to contemplate the meaning behind it as it is simply a definition.

Laplace Transform: Dirac Delta Function

Solve the second order ODE using method of Laplace Transform,

$$y'' + 2y' = \delta(t-1), \qquad y(0) = 0, \qquad y'(0) = 1.$$

We begin by apply the Laplace transform

$$y'' + 2y' = \delta(t-1) \;\Rightarrow\; \mathcal{L}\{y'' + 2y' = \delta(t-1)\}$$

NOTE The Laplace Operator is a linear transformation.

DEFINITION **Linear Transformation – Laplace Operator is a Linear Transformation**

$$\mathcal{L}\{f(\alpha t) + g(\beta t)\} = \mathcal{L}\{f(\alpha t)\} + \mathcal{L}\{g(\beta t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$$

$$\Rightarrow\; \mathcal{L}\{y''\} + \mathcal{L}\{2y'\} = \mathcal{L}\{\delta(t-1)\} \;\Rightarrow\; \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} = \mathcal{L}\{\delta(t-1)\}$$

Now that we are at the linear transformation, we need the Laplace Transformation information.

THEREOM Laplace Transform

$$\mathcal{L}\{y\} = Y(s), \qquad \mathcal{L}\{y'\} = sY(s) - y(0), \qquad \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0), \ldots$$

or

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

$$f, f', \ldots, f^{(n-1)} \text{ are continuous on } \mathbb{R}^+ \cup \{0\} \equiv [0, \infty).$$

$$\Rightarrow\; [s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] = \mathcal{L}\{\delta(t-1)\}$$

Dirac Delta Laplace Transform

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$\Rightarrow\ [s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] = e^{-st_0}$$

$$\Rightarrow\ s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) = e^{-st_0}\ [\text{solve for } Y(s)]$$

$$\Rightarrow\ s^2Y(s) + 2sY(s) - sy(0) - y'(0) - 2y(0) = e^{-st_0}$$

$$\Rightarrow\ s^2Y(s) + 2sY(s) = e^{-st_0} + \big(sy(0) + y'(0) + 2y(0)\big)$$

$$\Rightarrow\ s^2Y(s) + 2sY(s) = e^{-st_0} + \big((s)(0) + (1) + 2(0)\big)$$

$$\Rightarrow\ Y(s)(s^2 + 2s) = e^{-st_0} + 1$$

$$\Rightarrow\ Y(s) = \frac{e^{-s(1)} + 1}{s^2 + 2s} = \frac{e^{-s} + 1}{s^2 + 2s}.$$

Now, the tricky part: We have to arrange this in a form that fits a pattern that matches a preexisting Laplace Inverse. We go to the table of Laplace Inverses (usually in the back of the textbook).

Laplace Inverse

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \qquad \mathcal{L}^{-1}\{Y(s)\} = y(t).$$

Examine the possible alterations to the existing function before making any final decisions.

$$Y(s) = \frac{e^{-s} + 1}{s^2 + 2s} = \frac{e^{-s}}{s^2 + 2s} + \frac{1}{s^2 + 2s} = e^{-s}\frac{1}{s^2 + 2s} + \frac{1}{s^2 + 2s}$$

→TYPO TO BE CORRECTED STARTING HERE← [everything above is correct]

$$= e^{-s}\frac{1}{(s+1)^2 - 2} + \frac{1}{(s+1)^2 - 2} = e^{-s}\frac{1}{(s+1)^2 - \sqrt{2}^2} + \frac{1}{(s+1)^2 - \sqrt{2}^2}$$

From the Laplace Inverse Table

$$\mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2 - k^2}\right\} = e^{at}\sinh kt$$

*BIG TYPO ↘

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{Y(s) = \frac{e^{-s}+1}{s^2+2s}\right\} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{e^{-s}\frac{1}{(s+1)^2-\sqrt{2}^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-\sqrt{2}^2}\right\}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{e^{-s}\frac{1}{(s+1)^2-\sqrt{2}^2}\right\} + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s-(-1))^2-\sqrt{2}^2}\right\}$$

$$\Rightarrow y(t) = f(t-1)\mathcal{U}(t-1) + \frac{1}{\sqrt{2}}e^{-t}\sinh(\sqrt{2}t)$$

$$= \frac{1}{\sqrt{2}}e^{-(t-1)}\sinh\left(\sqrt{2}(t-1)\right)\mathcal{U}(t-1) + \frac{1}{\sqrt{2}}e^{-t}\sinh(\sqrt{2}t).$$

[FC] To prove that the answer is in fact the correct answer, take it and then compute the derivatives and plug them into $y'' + 2y' = \delta(t-1)$ (the original equation); if the equation balances—that is, $0 = 0$, then you know you did it correctly. Hence, you do not need ChatGPT or a teacher's solution manual and you have effectively done mathematics—not puzzle solving. The time to do this, is too lengthy at this moment so we will check the answer in the back of the book but before checking the answer, make sure you take the time to acknowledge the steps of reverse engineering and proving the solution you provided that outputted the answer, is in fact the correct answer.

NOTE This was an hour and half of work and it may not be right. We check the answer in the back of the book now.

The books answer is

$$y = \frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1).$$

This does not mean our answer is incorrect, it may be in a different form is all. Our answer is similar but not identical. Recall the hyperbolic trigonometric function

$$\sinh(t) = \frac{e^t - e^{-t}}{2}.$$

For this question, I will leave it to you to verify whether I made a typo or if

$$\frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1) = \frac{1}{\sqrt{2}}e^{-(t-1)}\sinh\left(\sqrt{2}(t-1)\right)\mathcal{U}(t-1) + \frac{1}{\sqrt{2}}e^{-t}\sinh(\sqrt{2}t).$$

Check

$$\mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2-k^2}\right\} = e^{at}\sinh kt = e^{at}\left[\frac{e^{kt}-e^{-kt}}{2}\right] \Rightarrow \sinh\left(\sqrt{2}t\right) = \frac{e^{\sqrt{2}t}-e^{-\sqrt{2}t}}{2}$$

There answer might be wrong or mine may be wrong or they are both the same. You check on your own without ChatGPT or solution manuals. Do not give up! We will come back to this question tomorrow and go back through it to be sure and finalize the answer and verify if it is correct after you all review it as part of the researchers.

→ ALL BELOW THIS LINE IS CORRECT → Corrected version below (reference both)

Starting with

$$Y(s) = \frac{e^{-s}+1}{s^2+2s} = \frac{e^{-s}+1}{s^2+2s+1-1} = \frac{e^{-s}+1}{(s+1)^2-1} = \frac{e^{-s}+1}{\left(s-(-1)\right)^2-1}$$

From the Laplace Inverse Table

$$\mathcal{L}^{-1}\{Y\} = y, \qquad \mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2-k^2}\right\} = e^{at}\sinh kt$$

$$\mathcal{L}^{-1}\{e^{-bs}F(s)\} = f(t-b)\mathcal{U}(t-b)$$

Note that the "a" and "b" are both a in the formula book but they are different a's here.

$$Y(s) = \frac{e^{-s}+1}{s^2+2s} = \frac{e^{-s}+1}{s^2+2s+1-1} = \frac{e^{-s}+1}{(s+1)^2-1} = \frac{e^{-s}+1}{\left(s-(-1)\right)^2-1}$$

$$= e^{-s}\frac{1}{\left(s-(-1)\right)^2-1} + \frac{1}{\left(s-(-1)\right)^2-1} = e^{-s}Y(s) + Y(s)$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{e^{-s}Y(s)\} + \mathcal{L}^{-1}\{Y(s)\}$$

$$\Rightarrow \mathcal{L}^{-1}\{e^{-s}Y(s)\} + \mathcal{L}^{-1}\{Y(s)\} = y(t-1)\mathcal{U}(t-1) + e^{-t}\sinh t$$

$$= e^{-(t-1)}\sinh(t-1)\,\mathcal{U}(t-1) + e^{-t}\sinh t$$

Hyperbolic Sine Function Operation

$$\sinh(t) = \frac{e^t-e^{-t}}{2}, \qquad \exp\{at\} = e^{at} = \sum_{k=0}^{\infty}\frac{a^kt^k}{k!} = 1 + at + \frac{a^2t^2}{2} + \cdots$$

$$= e^{-(t-1)}\left[\frac{e^{t-1}-e^{-(t-1)}}{2}\right]\mathcal{U}(t-1)+e^{-t}\left[\frac{e^{t}-e^{-t}}{2}\right]$$

$$= \left[\frac{e^{(t-1)-(t-1)}-e^{-(t-1)-(t-1)}}{2}\right]\mathcal{U}(t-1)+\frac{e^{t-t}-e^{-t-t}}{2}$$

$$= \left[\frac{e^{0}-e^{-2(t-1)}}{2}\right]\mathcal{U}(t-1)+\frac{e^{0}-e^{-2t}}{2}$$

$$= \left[\frac{1-e^{-2(t-1)}}{2}\right]\mathcal{U}(t-1)+\frac{1-e^{-2t}}{2}.$$

The books answer is

$$y=\frac{1}{2}-\frac{1}{2}e^{-2t}+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1).$$

[FC] Now, we will complete this solution by verifying our answer is indeed correct.

Before moving forward, you should challenge your mind to see if you yourself can figure out the method to verifying if the answer is correct or not.

[FCS] For this case: all we have to do is take the answer and plug it into the original equation and see if it balances to $0 = 0$.

We must take the derivative of y twice and plug it into

$$y''+2y'=\delta(t-1), \qquad y(0)=0, \qquad y'(0)=1.$$

NOTE Just because our answer matches the solution manual, doesn't guarantee the answer is correct. The TA that solved it may very well have made the same mistake we did. You need to know how to ensure your answers are correct without a preexisting answer and without technology before you are allowed to use technology.

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$$y=\frac{1}{2}-\frac{1}{2}e^{-2t}+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1)$$

$$\Rightarrow \frac{d}{dt}\left[y=\frac{1}{2}-\frac{1}{2}e^{-2t}+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1)\right] \Rightarrow \frac{d}{dt}y=\frac{d}{dt}\frac{1}{2}-\frac{1}{2}\frac{d}{dt}e^{-2t}+\frac{d}{dt}\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1)$$

$$\Rightarrow\ y' = 0 - \frac{1}{2}\left(e^{-2t}\frac{d}{dt}(-2t)\right) + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\frac{d}{dt}\mathcal{U}(t-1) + \mathcal{U}(t-1)\frac{d}{dt}\left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]$$

From later on, we come back to this (skip for now and come back).

$$y' = -\frac{1}{2}\left(e^{-2t}\frac{d}{dt}(-2t)\right) + \mathcal{H}'(t)$$

$$\Rightarrow\ y' = -\frac{1}{2}\left(e^{-2t}(-2)\right) + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\frac{d}{dt}\mathcal{U}(t-1) + \mathcal{U}(t-1)\left[\frac{d}{dt}\frac{1}{2} - \frac{1}{2}\frac{d}{dt}e^{-2(t-1)}\right]$$

Right now, it is not necessary to take the derivative of the $\mathcal{U}$ as it should balance in operator form just fine.

$$\Rightarrow\ y' = -\frac{1}{2}\left(e^{-2t}(-2)\right) + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \mathcal{U}(t-1)\left[0 - \frac{1}{2}e^{-2(t-1)}\frac{d}{dt}(-2(t-1))\right]$$

$$\Rightarrow\ y' = e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \mathcal{U}(t-1)\left[-\frac{1}{2}e^{-2(t-1)}\left(-2\frac{d}{dt}(t-1)\right)\right]$$

$$\Rightarrow\ y' = e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \mathcal{U}(t-1)\left[-\frac{1}{2}e^{-2(t-1)}\left(-2\left(\frac{d}{dt}t - \frac{d}{dt}1\right)\right)\right]$$

$$\Rightarrow\ y' = e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \mathcal{U}(t-1)\left[-\frac{1}{2}e^{-2(t-1)}\left(-2\left(\frac{dt}{dt} - 0\right)\right)\right]$$

$$\Rightarrow\ y' = e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \mathcal{U}(t-1)\left[-\frac{1}{2}e^{-2(t-1)}(-2(1))\right]$$

$$\Rightarrow\ y' = e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \mathcal{U}(t-1)\left[e^{-2(t-1)}\right]$$

Now, we need y'' to balance the second-order-linear-nonhomogeneous-ODE

NOTE "homogeneous" is pronounced home-ah-gene-E-us.

[DEF] **Homogenization** (1) a process by which the fat droplets from milk are emulsified and the cream does not separate. (2) the process of making things uniform or similar.

It ain't be no poe tait toe / po taht toe because der aint no Z up inside dat "homogeneous."

$$\Rightarrow\ \frac{d}{dt}\left[y' = e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \mathcal{U}(t-1)\left[e^{-2(t-1)}\right]\right]$$

$$\Rightarrow\ \frac{d}{dt}y' = \frac{d}{dt}e^{-2t} + \frac{d}{dt}\left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \frac{d}{dt}\mathcal{U}(t-1)\left[e^{-2(t-1)}\right]$$

To save time, recognize that one piece in the second derivative is exactly the same as the first derivative—that is,

$$\frac{d}{dt}\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) = \left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\frac{d}{dt}\mathcal{U}(t-1) + \mathcal{U}(t-1)\frac{d}{dt}\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right] = \mathcal{H}'(t).$$

We don't actually need to expand ↑ because the two objects from y' and y'' will cancel regardless of the expansion. So, to save writing, let us label the above as $\mathcal{H}'(t)$.

$$\frac{d}{dt}\mathcal{U}(t-1)\left[e^{-2(t-1)}\right] = \mathcal{U}(t-1)\frac{d}{dt}\left[e^{-2(t-1)}\right] + \left[e^{-2(t-1)}\right]\frac{d}{dt}\mathcal{U}(t-1)$$

$$= \mathcal{U}(t-1)\left(-2e^{-2(t-1)}\right) + \left[e^{-2(t-1)}\right]\mathcal{U}'(t-1), \qquad \frac{d}{dt}e^{-2t} = -2e^{-2t}.$$

$$\therefore y'' = -2e^{-2t} + \mathcal{H}'(t) + \mathcal{U}(t-1)\left(-2e^{-2(t-1)}\right) + \left[e^{-2(t-1)}\right]\mathcal{U}'(t-1),$$

$$y' = e^{-2t} + \mathcal{H}'(t).$$

Assuming our derivatives are correct, we will plug and chug! For those of you watching or reading in the future, if there is a typo after we plug this in, then your job will be to find it, and put it on the website to get credit for the book in the research position certification.

$$y'' + 2y' = \delta(t-1)$$

To show something is correct, you start on the left side of the equation and move until it equals the right side of the equation.

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$$\therefore\ y'' + 2y' = \left[-2e^{-2t} + \mathcal{H}'(t) + \mathcal{U}(t-1)\left(-2e^{-2(t-1)}\right) + \left[e^{-2(t-1)}\right]\mathcal{U}'(t-1)\right] + 2\left[e^{-2t} + \mathcal{H}'(t)\right]$$

$$= -2e^{-2t} + \mathcal{H}'(t) + \mathcal{U}(t-1)\left(-2e^{-2(t-1)}\right) + \left[e^{-2(t-1)}\right]\mathcal{U}'(t-1) + 2e^{-2t} + 2\mathcal{H}'(t)$$

$$= -2e^{-2t} + 2e^{-2t} + \mathcal{H}'(t) - 2e^{-2(t-1)}\mathcal{U}(t-1) + e^{-2(t-1)}\mathcal{U}'(t-1) + 2\mathcal{H}'(t)$$

$$= 3\mathcal{H}'(t) - 2e^{-2(t-1)}\mathcal{U}(t-1) + e^{-2(t-1)}\mathcal{U}'(t-1)$$

$$= 3\mathcal{H}'(t) - 2e^{-2(t-1)}[\mathcal{U}(t-1) - \mathcal{U}'(t-1)]$$

$$= 3\left[\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \mathcal{U}(t-1)\left[e^{-2(t-1)}\right]\right] - 2e^{-2(t-1)}[\mathcal{U}(t-1) - \mathcal{U}'(t-1)]$$

$$= 3\left[\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \mathcal{U}(t-1)\left[e^{-2(t-1)}\right]\right] - 2e^{-2(t-1)}[\mathcal{U}(t-1) - \mathcal{U}'(t-1)]$$

$$= 3\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + 3\mathcal{U}(t-1)\left[e^{-2(t-1)}\right] - 2e^{-2(t-1)}\mathcal{U}(t-1) + 2e^{-2(t-1)}\mathcal{U}'(t-1)$$

$$= 3\left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)} + 2e^{-2(t-1)}\right]\mathcal{U}'(t-1) + e^{-2(t-1)}\mathcal{U}(t-1)$$

$$\stackrel{?}{=} \delta(t-1)$$

Recall the **definition of the Dirac Delta Function**. We are going to reverse it.

$$\lim_{a\to 0}\delta_a(t-t_0) = \delta_0(t-t_0) \equiv \delta(t-t_0).$$

$$\text{[i]}\ \ \delta(t-t_0) = \begin{cases}\infty, & t = t_0\\ 0, & t \neq t_0\end{cases}, \qquad \text{[ii]}\ \int_0^\infty \delta(t-t_0)\,dt = 1.$$

It could be right. But what is $\mathcal{U}$?

The Unit Step Function

$$\mathcal{U}(t-a) = \begin{cases}0, & 0 \le t < a,\\ 1, & t \ge a.\end{cases}$$

$$= 3\left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)} + 2e^{-2(t-1)}\right]\mathcal{U}'(t-1) + e^{-2(t-1)}\mathcal{U}(t-1)$$

MISTAKE ↓

The million-dollar question: What is $\mathcal{U}' = ?$

NOTE Had we acknowledged the "Unit Step Function" first. We would have saved ourselves a lot of tedious calculation.

$$y' = e^{-2t} + \frac{1}{2}\frac{d}{dt}\mathcal{U}(t-1) - \frac{1}{2}\frac{d}{dt}e^{-2(t-1)}\mathcal{U}(t-1)$$

$$= e^{-2t} + \frac{1}{2}\frac{d}{dt}\mathcal{U}(t-1) - \frac{1}{2}\left(e^{-2(t-1)}\frac{d}{dt}\mathcal{U}(t-1) + \mathcal{U}(t-1)\frac{d}{dt}e^{-2(t-1)}\right).$$

Since the Unit Step Function is a piecewise-constant function, where its derivative is 0 everywhere except when $t = t_0$. Hence,

$$y' = e^{-2t} + e^{-2(t-1)}\mathcal{U}(t-1)$$

That looks a heck of a lot easier to work with!

$$y'' = \frac{d}{dt}e^{-2t} + \frac{d}{dt}e^{-2(t-1)}\mathcal{U}(t-1)$$

$$\Rightarrow \ \ y'' = -2e^{-2t} + e^{-2(t-1)}\frac{d}{dt}\mathcal{U}(t-1) + \mathcal{U}(t-1)\frac{d}{dt}e^{-2(t-1)}$$

$$\Rightarrow \ \ y'' = -2e^{-2t} - 2e^{-2(t-1)}\mathcal{U}(t-1)$$

Check answer

$$y'' + 2y' = \left[-2e^{-2t} - 2e^{-2(t-1)}\mathcal{U}(t-1)\right] + 2\left[e^{-2t} + e^{-2(t-1)}\mathcal{U}(t-1)\right]$$

$$= -2e^{-2t} - 2e^{-2(t-1)}\mathcal{U}(t-1) + 2e^{-2t} + 2e^{-2(t-1)}\mathcal{U}(t-1)$$

$$= 0.$$

So, does

$$0 = \delta(t-1)?$$

Recall part (i) of the definition of the Dirac Delta function.

$$\delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases}.$$

There are two t_0. From the initial conditions, we have $y(t_0) = y_0,\ y'(t_0) = y_0'$ where $t_0 = 0$.

We then have $\delta(t - t_0)$ where $t_0 = 1$. These are two different conditions not to be mixed up just like the a, b in the exponent earlier with the transform formulae.

INCORRECT BELOW ↓

It would appear, everywhere $t \neq 1$, the solution is correct. Since this is an initial condition problem, where we have $y(0) = 0, y'(0) = 1$. This tells us $t_0 = 0.$ And since $t_0 = 1 \neq 0$. $\delta(t - t_0) = 0$. Banger!

Yes, the answer is $0 = 0$. It balances. This shows that our solution was correct because the answer balanced the equation. That still does not mean it is 100% guaranteed. We would require an eye from team of people to fully concur that it is correct. Even still, it may not be correct. A famous paper that knighted Andrew Wiles for proving Fermat's Last Theorem, turned out to have a big typo and it was reviewed by many peers. It was still correct but wasn't.

INCORRECT ABOVE ↑

In the above, I has stated that the t_0 is the initial condition from the IVP but the actual t_0 is coming from the Dirac Delta condition $\delta(t-1)$ where $t_0 = 1$.

Now, from $0 = \delta(t-1)$, we know that $t \in [0, \infty)$, $t_0 > 0$ from the initial definitions.

So, for the Dirac Delta,

$$\delta(t - t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases},$$

We can say with confidence $0 = \delta(t-1)$ is true iff $t \neq 1$. Thus, the correct answer for this question should be

$$y(t) = e^{-(t-1)} \sinh(t-1)\, \mathcal{U}(t-1) + e^{-t} \sinh t\,, \qquad t \in [0,1) \cup (1, \infty).$$

Is this 100% correct statement?