NOTE Linear Algebra is the study of vector space; matrices are just tools or notations to simplify calculations more or less....

We will start by organizing notation.

Trig/Physics/3D CALC	Linear Algebra	Higher Math (in general)
$\vec{v} = \langle x, y, z\rangle$	$\mathbf{x} = (x_1, x_2, x_3, \dots)$	$x \equiv \vec{x} \equiv \mathbf{x} = (x_1, x_2, x_3, \dots)$

The above vector notations are all the same. In Quantum mechanics, the column vector is $v = |x, y, z\rangle$. All vectors can be considered column vectors in the context of linear algebra vector and matrix notation. Now, let's use the notation to assemble the matrix notation.

Solve $ax + by = c$ and $dx + ey = f$ by putting them into standard matrix equations.

First, I will index everything. $a_{11}x_1 + a_{12}x_2 = b_1$ and $a_{21}x_1 + a_{22}x_2 = b_2$.

Recall the "dot product" and(or) "inner product," $(a, b) \cdot (c, d) = ac + bd = (a, b)^T(c, d)$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \Rightarrow \begin{aligned} (a_{11} \quad a_{12})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= b_1 \\ (a_{21} \quad a_{22})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= b_2 \end{aligned} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Coefficient Matrix	**Variable Vector**	**Constant Vector**
$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$	$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

The above titles may change chapter to chapter

The matrix equation, $A\mathbf{x} = \mathbf{b}$	Augmented Matrix, $[A\|\mathbf{b}]$
$A\mathbf{x} = \mathbf{b} \Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$	$[A\|\mathbf{b}] \Leftrightarrow \left[\begin{array}{cc\|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$

The span of set of standard column vectors is the **linear Combination** which ultimate equals $A\mathbf{x}$.

Generalized Definition "Span of Set" (changes book to book)

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots\} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots = \mathbf{v}_1x_1 + \mathbf{v}_2x_2 + \cdots = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = A\mathbf{x}.$$

From the above, we would see that $\mathbf{v}_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$. The column vectors of A.

Show that the span of $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is $A\mathbf{x}$.

$$\text{span}\mathcal{V} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_1 x_1 + \mathbf{v}_2 x_2 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} x_2$$

$$= \left[\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \mid \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{x} = A\mathbf{x}.$$

Other forms:

$$\begin{pmatrix} a_{11}x_1 \\ a_{21}x_1 \end{pmatrix} + \begin{pmatrix} a_{12}x_2 \\ a_{22}x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{bmatrix} = \cdots$$

Linear Algebra is most definitely the art of giving the same thing different names. You will be asked to find the same solution over and over again, but it will be worded differently, and you will want the final answer stated a certain way, even though all the math/arithmetic is the same. E.g., finding the kernel, nullspace, solution to $A\mathbf{x} = \mathbf{0}$, the orthogonal matrix, ... are all the same solution methods but are written differently, such as a vector solution, set solution, general solution, parametric solution, ... All the same arithmetic and numbers but written differently. **In this subject, all vectors are considered column vectors unless they are specified to be row vectors or are transposed**. Transposing a matrix (vector) turns a row/column into a column/row.

"Mathematics is the art of giving the same name to different things."—Henri Poincaré

