The Kronecker Delta

[12]

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

What is the Kronecker delta? It is a scalar or “Rank Zero Tensor” in “Tensor Analysis.” Really though, it is the dot product of unit vectors.

This is easy enough to prove (or derive) in nth order or 3D space.

Let $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)$ or $\mathbf{e}_{\boldsymbol{n}+1} = (0, \ldots 1, \ldots 0)$. To begin, let’s stick with 3D space as the majority of you will be in physics/engineering and will not be concerned just yet with higher order math.

For the ‘Dot Product,’ consider

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (1)(0) + (0)(1) + (0)(0) = 0,$$

$$\mathbf{e}_3 \cdot \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (0)(0) + (0)(0) + (1)(1) = 1.$$

It is easy to conclude, $\mathbf{e}_i \cdot \mathbf{e}_j = 1 \Leftrightarrow i = j$ and $\mathbf{e}_i \cdot \mathbf{e}_j = 0 \Leftrightarrow i \neq j$. Hence, $\delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$.

This is a very basic connection, but we should take it a step further to differentiate notations course-to-course.

For the ‘Inner Product,’ consider

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \mathbf{e}_i^T \mathbf{e}_j = \mathbf{e}^i \mathbf{e}_j = \delta_j^i \quad \text{or} \quad \langle \mathbf{e}_i | \mathbf{e}_j \rangle \equiv \mathbf{e}_i \cdot \mathbf{e}_j \text{ [Quantum Mechanics]}$$

In 3D index/tensor math for physics and engineering, likely, you will use the dot/cross product subscript notations. In some higher physics/math, you may see the inner product style.

Assignment: Show that $\langle \mathbf{e}_i, \mathbf{e}_j \rangle$ is equal to one or zero for nth dimensional space.

Deriving Dot and Inner Product of Vectors

Consider $\vec{u} = \langle u_1, u_2, u_3 \rangle,\ \vec{v} = \langle v_1, v_2, v_3 \rangle$. The

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = (u_1)(v_1) + (u_2)(v_2) + (u_3)(v_3) = u_1 v_1 + u_2 v_2 + u_3 v_3$$

We can see that the sum is

$$u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i.$$

This doesn't help much in our conquest. We must take it a different way involving unit vectors and the bracket vector notation falls apart in higher order. Going forward, we will omit 'dot product' and 'vector angle bracket notation' not to be confused with inner product angle bracket notation, $\langle \mathbf{u}, \mathbf{v} \rangle$. Recall $\mathbf{u} \equiv \vec{u} \equiv u$ depending on the course/subject/professor/book.

Let $\mathbf{u} = (u_1, u_2, \dots, u_n), \mathbf{v} = (v_1, v_2, \dots, v_n)$ be column vectors (always assumed). Continue now with inner product notation.

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v} = [u_1 \quad u_2 \quad \cdots \quad u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=n}^{n} u_i v_i.$$

This still doesn't do much for us so we will now move to extracting the unit vectors and then performing the inner product notation—that is,

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = v_1 \mathbf{e}_1 + v_2 \boldsymbol{e}_2 + \cdots + v_n \boldsymbol{e}_3$$

$$= \sum_{i=1}^{n} v_i \mathbf{e}_i. \quad \therefore \mathbf{u} = \sum_{i=1}^{n} u_i \mathbf{e}_i \equiv u_i \mathbf{e}_i \ \text{[Einstein Summation Convention]}$$

Now, we have a condensed vector or a rank 1 tensor (vector). Tensor Definitions:

Rank Zero Tensor [scalar]	Rank One Tensor [vector]	Rank Two Tensor [matrix]
Vector in $\mathbb{R}^n$ Scalar $n^0 = 1$.	Vector in $\mathbb{R}^n$ Vector with n^1 elements.	Matrix in $M_{n \times n}$ Square Matrix with n^2 elements

The index notation vs. tensor notation vs. tensor analysis is essentially the same only in tensor math, you are **renaming the objects** as tensors and the element's location will have another name. In a nutshell, the vector $\mathbf{v} = v_i \mathbf{e}_i$ is a rank one tensor.

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The Inner Product of nth Dimensional Space

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= (u_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix} + \cdots) \left(v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots \right)$$

We know from inner product notation, that the following 'foils.'

$$= u_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix} v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + u_n \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix} v_n \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

We can commute scalars.

$$= u_1 v_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + u_2 v_2 \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + u_n v_n \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$= u_1 v_1 \mathbf{e}^1 \mathbf{e}_1 + u_2 v_2 \mathbf{e}^2 \mathbf{e}_2 + \cdots + u_n v_n \mathbf{e}^n \mathbf{e}_n = \sum_{i=1}^{n} u_i v_i \mathbf{e}^i \mathbf{e}_i.$$

We can now say the dot product [form 1] is $\vec{u} \cdot \vec{v} = u_i v_i \mathbf{e}^i \mathbf{e}_i = \mathbf{u}^T \mathbf{v}$.

Dot (or Inner) Product [form 1]

$$\mathbf{u}^T \mathbf{v} = \vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i \mathbf{e}^i \mathbf{e}_i$$

When we get to the cross product and higher, we introduce double and triple summations. The dot or inner product may be better suited with multiple indices.

Assignment, show the following is true

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} \sum_{j=1}^{n} u_i v_j \delta_{ij}$$ [hint: add 0 to **form 1** many times].

The Inner (or dot) Product [form 2]

$$\vec{u} \cdot \vec{v} = \mathbf{u}^T \mathbf{v} = u_i v_j \delta_{ij}$$

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