Show that $\{e^x, e^{-x}, -e^x\}$ is a linearly dependent set.

$$\text{span}\,\{e^x, e^{-x}, -e^x\} = c_1e^x + c_2e^{-x} + c_3(-e^x) = c_1e^x + c_2e^{-x} + c_4e^x$$

$$= (c_1 + c_4)e^x + c_2e^{-x} = c_5e^x + c_2e^{-x} \quad \Rightarrow \quad \{e^x, e^{-x}, -e^x\} \sim \{e^x, e^{-x}\}.$$

Thus, the set $\{e^x, e^{-x}, -e^x\}$ is a linearly dependent set.

In general—that is, reference your textbook for the proper/formal definition/theorem(s).

For the above, if the solution to a differential equation is $y = c_1e^x + c_2e^{-x} + c_3(-e^x)$,

Then,

$$y = c_1e^x + c_2e^{-x} + c_3(-e^x) = c_1e^x + c_2e^{-x} + c_4e^x$$

$$= (c_1 + c_4)e^x + c_2e^{-x} = c_5e^x + c_2e^{-x} = [e^x \quad e^{-x}]\begin{bmatrix} C_2 \\ C_5 \end{bmatrix} = [e^x \quad e^{-x}]\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Also note: span $\{e^x, e^{-x}, -e^x\} = c_1e^x + c_2e^{-x} + c_3(-e^x)$.

If you are in a differential equations course with linear algebra, the techniques and definitions are going to change from your standard linear algebra courses.

NOTE At this part of the book, since I am not referencing a textbook, the notation would be classified as slang use. The point is to get you familiar with referencing multiple sources.

NOTE If you are prompted to see if something is within the span of a set, you simple set the span of the set equal to the object. Then, check if there is a unique solution.

E.g., is e^{2x} in the span $\{e^x, e^{-x}, -e^x\}$?

$$[e^x \quad e^{-x}]\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = [e^{2x}] \quad \Leftrightarrow \quad c_5e^x + c_2e^{-x} = e^{2x}$$

$$\Rightarrow \; [e^x \quad e^{-x}]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = (1 \times 2)(2 \times 2)(2 \times 1) = (1 \times 1) = e^{2x}$$

$$\Rightarrow \; [e^x \quad e^{-x}]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = [e^{2x} \quad *]\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

I.e. $e^{2x} \notin$ span $\{e^x, e^{-x}, -e^x\}$. There are no constants that could multiply $\{e^x, e^{-x}, -e^x\}$ any of these to make them equal. How come $c_1 = e^x$ making $c_1e^x = e^xe^x = e^{x+x} = e^{2x}$?—because that is not a constant, it is a varying function. The question does not make sense to begin with because it is like doing calculations between $\mathbb{R}^3$ and $\mathbb{R}^5$. They are not compatible.