10.PE.47 [7] Power Series – Sequences and Series in Calculus

Radius and Interval of Convergence

QUESTION

(a) Find the series' radius and interval of convergence. Then identify the values of x for which the series converges (b) absolutely and (c) conditionally.

$$\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n}$$

ANSWER

$$R = \sqrt{3}, \qquad I = \left(-\sqrt{3}, \sqrt{3}\right)$$

[**Step 1: Process of Elimination**] For the most part, the ratio or root test are the tests to be used for interval of converge. Since this as a power of $2n-1$, the root test may or may not be the best route. When doing intervals of convergence, a good fail safe, is to simple use the ratio test.

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} < 1, & Abs\text{Converges} \\ > 1, & \text{Diverges} \\ = 1, & \text{Inconclusive} \end{cases}.$$

The above is worth 50% of the credit (unless the test doesn't work).

[**Step 2: Execute the Formula**]

$$a_n = \frac{(n+1)x^{2n-1}}{3^n} \Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\big((n+1)+1\big)x^{2(n+1)-1}}{3^{(n+1)}}}{\dfrac{(n+1)x^{2n-1}}{3^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\big((n+1)+1\big)x^{2(n+1)-1}}{3^{(n+1)}} \cdot \frac{3^n}{(n+1)x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+2-1}}{3^n \cdot 3} \cdot \frac{3^n}{(n+1)x^{2n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+1}}{\cancel{3^n} \cdot 3} \cdot \frac{\cancel{3^n}}{(n+1)x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+1}}{3} \cdot \frac{1}{(n+1)x^{2n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+1}}{3} \cdot \frac{1}{(n+1)x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+1}}{3(n+1)x^{2n-1}}\right|$$

[PL] At this point, people often forget to keep the absolute value on the variable.

$$= \lim_{n\to\infty}\left|\frac{(n+2)x^{(2n+1)-(2n-1)}}{3(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+1-2n+1}}{3(n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)x^2}{3(n+1)}\right| = \left|\frac{x^2}{3}\right|\lim_{n\to\infty}\frac{(n+2)}{(n+1)} = \left|\frac{x^2}{3}\right|\lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}\frac{n}{n}$$

$$= \left|\frac{x^2}{3}\right|\frac{\left(\lim_{n\to\infty}1+\lim_{n\to\infty}1\frac{2}{n}\right)}{\left(\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{1}{n}\right)} = \left|\frac{x^2}{3}\right|\frac{(1+0)}{(1+0)} = \left|\frac{x^2}{3}\right| < 1.$$

In order for this to be convergent, $\left|\frac{x^2}{3}\right| < 1$. Solve the inequality.

$$\left|\frac{x^2}{3}\right| < 1 \;\Rightarrow\; |x^2| < 3 \;\Rightarrow\; x < \sqrt{3} \text{ or } x > -\sqrt{3}$$

$$\Rightarrow\; x < \sqrt{3}, \qquad x > \sqrt{-3}.$$

[PL] At this point, students always forget to test the endpoints. The answer does not include the endpoints. It was chosen to specifically set you up for failure. Why? Because, when the exam comes, you will need to test the endpoints and one, both or neither will be included. If you don't show this, you won't get full credit.

Endpoint $x = -\sqrt{3}$

$$\sum_{n=0}^{\infty}\frac{(n+1)x^{2n-1}}{3^n} \;\Rightarrow\; \sum_{n=0}^{\infty}\frac{(n+1)\left(-\sqrt{3}\right)^{2n-1}}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^{2n-1}(n+1)(3)^{n-1/2}}{3^n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{2n-1}(n+1)3^n}{\sqrt{3}3^n} = \sum_{n=0}^{\infty}\frac{(-1)^{2n-1}(n+1)}{\sqrt{3}} = \frac{1}{\sqrt{3}}\sum_{n=0}^{\infty}(-1)^{2n-1}(n+1)$$

We need to show this is divergent by using another test.

By the nth term for divergence, we see that the limit $(-1)^{2n-1}(n+1)$ does not exist. Thus, the series is divergent, and the endpoint is not included.

Endpoint $x = \sqrt{3}$

$$\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n} \;\Rightarrow\; \sum_{n=0}^{\infty} \frac{(n+1)\left(\sqrt{3}\right)^{2n-1}}{3^n} = \sum_{n=0}^{\infty} \frac{(n+1)(3)^{n-1/2}}{3^n}$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)}{\sqrt{3}} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} (n+1)$$

This is also divergent because the limit to infinity of $(n+1)$ is infinity or does not exist. Thus, by the nth term for divergence test, the series diverges and it is not included in the endpoint.

[**Step 3- Finalize**]

For part (a), we can clearly see that the radius is $R = \sqrt{3}$ since it is centered at the origin. And the interval is an open interval because the endpoints are divergent. Thus, $I = \left(-\sqrt{3}, \sqrt{3}\right)$.

For part (b), from the theorem of ratio test, we know that it is absolutely convergent for all values less than 1. Thus, it is absolutely convergent on the interval of convergence.

A series is conditionally convergent, when the series is convergent and

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Conditional Convergence

A convergent series that is **not absolutely convergent** is conditionally convergent.

Since the series is convergent only on the interval found, and by definition of ratio test, it is also absolutely convergent on that interval, meaning it is not conditionally convergent for any x.

[R.P.] For those in the research position, you can now ponder and review to find any issue or error. We will reconnect to finalize in the next session.